Solving the relations (1) and (2), we get

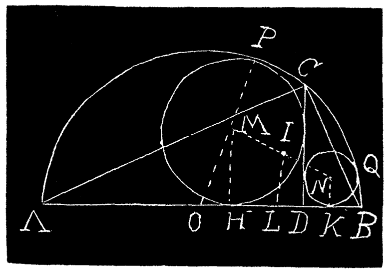
$$x_1 = y_1 = -k - r + \sqrt{(2r^2 + 2rk)},$$
$$x_{11} = -y_{11} = -r + k + \sqrt{(2r^2 - 2rk)}.$$

The x's represent the radii of the three circles, and satisfy the required conditions $x = \frac{x_1 + x_{11}}{2}$, and the centers are in a straight line since they satisfy the condition

$$\frac{y - y_1}{x - x_1} = \frac{y_{11} - y_1}{x_{11} - x_1}.$$

II. Solution by A. M. HARDING, A. M., University of Arkansas, and the PROPOSER.

Let M be the center of circle touching arc AC, sides AD and DC, and let H be the point where it touches AD. Let N be center of circle touching arc CB, sides CD and DB, and K the point where it touches DB. Also let I be center of circle inscribed in triangle ABC. $MH = HD$ since $\angle MDH = 45°$, and similarly, $NK = DK$. Therefore $HM + NK = HK$.



Again, by a well known theorem, $BC = BH$. Hence $\angle HCB = \angle CHB = \angle CAH + \angle ACH$, but $\angle BCD = \angle CAH$; therefore, $\angle HCD = \angle HCA$. Hence $\angle DAI + \angle ACH = 45°$. Again, $\angle ACI = 45°$, therefore $\angle DAI = \angle HCI$. Hence, the four points A, C, I, and H are concyclic. Hence, $\angle IHD = \angle ACI = 45°$. In a similar manner it can be shown that $\angle IKD = 45°$.

Now let L be the point where the circle inscribed in triangle ABC touches AB; then since $\angle ILH = 90°$, and $\angle IHL = \angle IKL = 45°$, line $IL = HL$ and $IL = LK$; but IL is the radius of circle inscribed in triangle ABC. Therefore $IL = \frac{1}{2}HK = \frac{1}{2}(MH + NK)$ by above.

Again, since L is the mid-point of HK, and IL, which $= \frac{1}{2}(MH + NK)$, is also parallel to MH and NK, it is easily seen that I is the mid-point of the line MN.

Also solved similarly by J. Scheffer.

CALCULUS.

318. Proposed by JOHN C. GREGG, Greencastle, Ind.

A thread is wound spirally n times around a cone, the radius of whose base is r, and slant height h, the turns being at uniform distance apart. If the thread is kept taut, what will be the length of the trace of its end on a horizontal plane?

I. Solution by B. F. FINKEL, Ph. D., Drury College.

A solution of this problem, by Henry Gunder, of Findlay (Ohio) College, was published in Vol. 9, page 199, of the *School Visitor*, published by John S. Royer. Professor Gunder's solution is wholly incorrect. So far as we know, no correct solution of the problem has ever been published. The problem as stated is indeterminate; for the reason that a string stretched taut on the surface of a cone cannot be in equilibrium unless held in position by the friction of the surface of the cone, by an adhesive substance, or else by pegs driven into the cone normal to the plane formed by the tangent to the string and the corresponding element of the cone. In the case of friction holding the string in position, it could not be wound on the cone nor unwound from it by merely taking hold of one end of the string and stretching it taut. Were a surface formed by conceiving a normal to the tangent of the string and the element of the cone, to move along the string, then the string could be wound on the cone by taking hold of one end of it and keeping it taut. The string would come in contact with this surface above the surface of the cone, and as the winding proceeds, the string would slip on this surface until it came in contact with the surface of the cone. The string would thus lie in the groove formed by the surface of the cone and the surface generated by a normal moving along the curve, formed by the proposed position of the string.

Since the unwinding as required in the problem is impossible, the problem is impossible of solution. If we suppose the string to adhere slightly to the cone, there are a number of ways which might be proposed to unwind the string. Of these, two would naturally suggest themselves. We might suppose, first, the string unwound in such a manner that the tangent to the projection of the string on the plane of the base and the unwound portion of the string are in the same plane; and second, that the unwound portion of the string and the altitude of the cone lie in the same plane. Both of these assumptions lead to very complicated computations. We shall now show, by a different method, what Professor Gunder obtained as a result, and also indicate a method of solution according to the two assumptions made above.

Let $C-AD'DBA$ be the cone, whose radius $OB=R$ and altitude $CO=h$; P, any point on the string, whose coördinates are (x, y, z); P', a consecutive point; $OP_1=\rho$, and $\angle BOP_1=\theta$. Draw the lines CPD, $CP'D'$, OD, OD', O_1P, and O_1K. P_1 is the projection of P; P_1' of P', and K_1 of K. P'_1P_1B is the projection of the string on the xy-plane; K_1P_1 is the projection of KP, and P'_1P_1 is the projection of IK and $P'P$.

The equation of the curve of the string is obtained by setting up a relation between $(z, \rho, \theta, n, R, h)$ or $(x, y, z, \theta, n, R, h)$.

In the similar triangles DP_1P and DOC, we have $R-\rho : R=z : h$, or

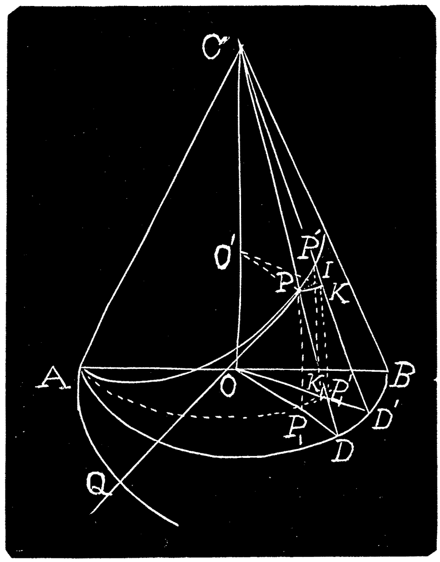
$$\rho=\frac{R(h-z)}{h}\dots(1).$$

Also, since the string makes n equidistant turns, we have arc DB : $2\pi Rn = z:h$, or $R\theta : 2\pi nR = z:h$, or

$$z = \frac{\theta h}{2\pi n} \ldots (2); \quad \therefore \rho = \frac{R}{2\pi n}(2\pi n - \theta) \ldots (3).$$

the equation of the projection of the curve of the string on the xy-plane.

Also, $x = \rho\cos\theta = \frac{R}{h}(h-z)\cos\theta$ $= \frac{R}{2\pi n}(2\pi n - \theta)\cos\theta \ldots (4)$; $y = \rho\sin\theta$ $= \frac{R(h-z)}{h}\sin\theta = \frac{R}{2\pi n}(2\pi n - \theta)\sin\theta$ $\ldots (5)$, $z = \frac{\theta h}{2\pi n}$ are the parametric equations of the curve of the string. The equations of the tangent to the string are:



$$\frac{\alpha - x}{\frac{dx}{ds}} = \frac{\beta - y}{\frac{dy}{ds}} = \frac{\gamma - z}{\frac{dz}{ds}}; \text{ or}$$

$$\frac{\alpha - x}{-\frac{R}{2\pi n}[\cos\theta + (2\pi n - \theta)\sin\theta]}$$

$$= \frac{\beta - y}{-\frac{R}{2\pi n}[\sin\theta - (2\pi n - \theta)\cos\theta]} = \frac{\gamma - z}{\frac{h}{2\pi n}} \ldots (6).$$

The coördinates of the point of intersection of the tangent with the xy-plane are found by letting $\gamma = 0$ in (6). Hence,

$$\alpha = \frac{R}{2\pi n}[(2\pi n - \theta)(\cos\theta + \theta\sin\theta) + \theta\cos\theta] \ldots (7), \text{ and}$$

$$\beta = \frac{R}{2\pi n}[(2\pi n - \theta)(\sin\theta - \theta\cos\theta) + \theta\sin\theta] \ldots (8).$$

The length of the curve described by the point of intersection of the tangent with the xy-plane is

$$s=\int\sqrt{[d\alpha^2+d\beta^2]}=\frac{R}{2\pi n}\int_0^{2\pi n}\theta\sqrt{[4+(2\pi n-\theta)^2]}d\theta$$

$$=\frac{4R}{3\pi n}+\tfrac{2}{3}R\left(\pi n-\frac{2}{\pi n}\right)\sqrt{[1+\pi^2n^2]}+2R\log[\pi n+\sqrt{(1+\pi^2n^2)}].$$

This is the result obtained by Professor Gunder in the solution referred to above, and cannot be correct unless the string, in unwinding, is "taken up" at the same time; for the portion of the string unwound at any time is longer than the tangent to the string at the point of contact with the cone. The length of a portion of the string is

$$l=\int\sqrt{[dx^2+dy^2+dz^2]}=\frac{1}{2\pi n}\int_0^{\theta}\sqrt{[h^2+R^2(2\pi n-\theta)^2]}d\theta$$

$$=\frac{1}{2\pi n}\Bigg[-\frac{R}{2}(2\pi n-\theta)\sqrt{\frac{h^2+R^2}{R^2}+(2\pi n-\theta)^2}-R\left(\frac{h^2+R^2}{R^2}\right)\log\left[(2\pi n-\theta)\right.$$

$$\left.+\sqrt{\frac{h^2+R^2}{R^2}+(2\pi n-\theta)^2}\right]+\pi n R\sqrt{\frac{h^2+R^2}{R^2}+4\pi^2n^2}$$

$$+R\left(\frac{h^2+R^2}{R^2}\right)\log\left(2\pi n+\sqrt{\frac{h^2+R^2}{R^2}+4\pi^2n^2}\right)\Bigg]$$

$$=\frac{1}{2\pi n}\Bigg[\pi n\sqrt{[h^2+R^2+4\pi^2n^2R^2]}-\tfrac{1}{2}\sqrt{[2\pi n-\theta]}\sqrt{[h^2+R^2+(2\pi n-\theta)^2R^2]}$$

$$+R\left(\frac{h^2+R^2}{R^2}\right)\log\left(\frac{2\pi nR+\sqrt{[h^2+R^2+4\pi^2n^2R^2]}}{[2\pi n-\theta]R+\sqrt{[h^2+R^2+R^2(2\pi n-\theta]^2}}\right)\Bigg].$$

The length of the corresponding tangent is

$$s=\sqrt{[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}=\frac{\theta}{2\pi n}\sqrt{[h^2+R^2(2\pi n-\theta)^2]}.$$

A comparison of these two expressions shows that the length of the unwound portion of the string is longer than the tangent to the point of contact. This fact also shows that the string cannot be unwound from the cone in the manner required by the problem unless it is held in contact with the cone by means of some force, as, for example, an adhesive substance.

Since the string must be held in position on the cone, we may choose any number of ways of unwinding it.

Of these ways, we have already mentioned two. First, suppose the string is unwound in such a way that the projection of the string on the xy-plane is the tangent to the curve $P_1'P_1B$. The equation of the trace of the string when unwound from the cone, by keeping it taut, is

$$OQ=[P_1Q^2+OP_1^2-2P_1Q.OP\cos\angle OP_1Q]^{\frac{1}{2}},$$

where $P_1Q=\sqrt{[PQ^2-P_1P^2]}=\sqrt{[s^2-z^2]}$, s being the unwound portion of the string, and $\angle QP_1O=\tan^{-1}[\rho\, d\,\theta/d\,\rho]$; or

$$OQ=\sqrt{s^2-z^2+\rho^2-\frac{2\rho\sqrt{[s^2-z^2]}}{[1+(2\pi n-\theta)^2]^{\frac{1}{2}}}}.$$

From this the length of the path may be found by approximate methods.

Second, suppose that the string is unwound so that the unwound part lies always in a plane with the axis of the cone. In this case the equation of the trace is $OQ=\sqrt{[s^2-z^2]}+\rho$, where s is the unwound portion of the string, and $\rho=OP_1$. The length of the trace in this case is

$$l=\int_0^{2\pi n}\{[(sds-zdz)(s^2-z^2)^{-\frac{1}{2}}+d\,\rho]^2+[\sqrt{(s^2-z^2)}+\rho]^2d\,\theta^2\}^{\frac{1}{2}}.$$

A solution, by approximate quadrature, may be completed in a manner similar to that indicated in my solution of problem 309, pages 107-110, Vol. XIII of MONTHLY.

II. Solution by PROFESSOR F. L. GRIFFIN, Reed College, Portland, Oregon.

I. *Preliminary remark.* The problem is impossible in the absence of friction; since a stretched string can be in equilibrium on a "smooth" surface only along a geodesic line, and no geodesic line has the proposed location. Choosing the axis of the cone as the Z-axis, and the vertex as the origin, the equation of the surface is

$$f(x,\ y,\ z)\equiv x^2+y^2-z^2\tan^2\alpha=0,$$

where α is the half-angle of the cone, or $\arctan(r/h)$. The differential equations of a geodesic are

$$\frac{d^2x}{ds^2}=\nu\frac{\partial f}{\partial x}=2\nu x,\ \ \frac{d^2y}{ds^2}=\nu\frac{\partial f}{\partial y}=2\nu y,\ \ \frac{d^2z}{ds^2}=\nu\frac{\partial f}{dz}=-2\nu\, z\tan^2\alpha,$$

where s denotes the length of the arc and

$$\nu=\sqrt{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2}\Big/\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}.$$

Combining the first pair of equations gives an integral, $x\frac{dy}{ds}-y\frac{dx}{ds}=c$, constant. But since for any point (x, y, z) on the cone, $\sqrt{[x^2+y^2]}=z\tan\alpha$, we have $x=z\tan\alpha\cos\phi$, $y=z\tan\alpha\sin\phi$, where ϕ denotes the polar angle of the projection of (x, y, z) on the horizontal reference plane. Then

$$\frac{dx}{ds}=\tan\alpha\left[\frac{dz}{ds}\cos\phi-z\sin\phi\frac{d\phi}{ds}\right],\quad \frac{dy}{ds}=\tan\alpha\left[\frac{dz}{ds}\sin\phi+z\cos\phi\frac{d\phi}{ds}\right],$$

whence the integral becomes $z^2\tan^2\alpha(d\phi/ds)=c$. (If $c=0$, then either $z=0$ permanently or else ϕ=constant, showing,— as is physically obvious,— that a thread cannot be thus wound up beginning at the vertex). Further,

$$1=\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=\tan^2\alpha\left[\left(\frac{dz}{ds}\right)^2+z^2\left(\frac{d\phi}{ds}\right)^2\right]+\left(\frac{dz}{ds}\right)^2$$

$$=\left[\left(\frac{dz}{d\phi}\right)^2\sec^2\alpha+z^2\tan^2\alpha\right]c^2/z^4\tan^4\alpha.$$

Separating variables, we have

$$\frac{dz}{z\sqrt{[z^2-c^2\cot^2\alpha]}}=-\frac{\sin\alpha\tan\alpha}{c}d\phi,$$

whence, k being constant. $z=c\cot\alpha\sec(kc\cot\alpha-\phi\sin\alpha)$.

Evidently, for small enough values of $\sin\alpha$ the geodesic may wind several times around the cone before receding to infinity at a certain finite value of ϕ; but even these few turns are never at uniform distance apart. For, if adding 2π to ϕ merely adds some constant to z, it follows that $(dz/d\phi)$ is the same at $\phi+2\pi$ as at ϕ, since

$$\frac{z(\phi+\triangle\phi+2\pi)-z(\phi+2\pi)}{\triangle\phi}\equiv\frac{z(\phi+\triangle\phi)-z(\phi)}{\triangle\phi};$$

but clearly the period of $dz/d\phi$ is $2\pi/\sin\alpha$.

II. But if we waive this question, and suppose the surface of the cone

sufficiently rough to hold the thread in the proposed position; or if we think of the equivalent purely geometric problem, the required length, σ, of the trace of the end-point (ξ, η, ς), may be found, or at least expressed as a definite integral, as follows:

The requirement $z(\phi+2\pi)-z(\phi)\equiv m$, constant, does not define z as a function of ϕ, but is most simply satisfied by $z=h-m\phi/2\pi$, or $\frac{dz}{d\phi}=-\frac{m}{2\pi}$. This gives, retaining z for brevity:

$$\frac{dx}{d\phi}=\tan\alpha\left[-\frac{m}{2\pi}\cos\phi-z\sin\phi\right],\quad \frac{d^2x}{d\phi^2}=\tan\alpha\left[\frac{m}{\pi}\sin\phi-z\cos\phi\right],$$

$$\frac{dy}{d\phi}=\tan\alpha\left[-\frac{m}{2\pi}\sin\phi+z\cos\phi\right],\quad \frac{d^2y}{d\phi^2}=\tan\alpha\left[-\frac{m}{\pi}\cos\phi-z\sin\phi\right],$$

$$\left(\frac{ds}{d\phi}\right)^2=z^2\tan^2\alpha+\frac{m^2}{4\pi^2}\sec^2\alpha,\quad \frac{d^2s}{d\phi^2}=z\tan^2\alpha\left(-\frac{m}{2\pi}\right)\Big/\frac{ds}{d\phi}.$$

From $ds/d\phi$ it is easy to obtain s, the length of the curve on the cone to any point, or, in particular, to $\phi=2\pi n$, which gives the total length, l, of the string. Now we find from the equation of the tangent at any point (x, y, z) of this spiral curve:

$$\frac{\xi-x}{\frac{dx}{ds}}=\frac{\eta-y}{\frac{dy}{ds}}=\frac{\varsigma-z}{\frac{dz}{ds}}=l-s=\text{length yet unwound.}$$

Hence, $\varsigma=x+(l-s)\frac{dx}{d\phi}\Big/\frac{ds}{d\phi}$, $\frac{d\xi}{d\phi}=(l-s)\dfrac{\frac{ds}{d\phi}\cdot\frac{d^2x}{d\phi^2}-\frac{dx}{d\phi}\cdot\frac{d^2s}{d\phi^2}}{\left(\frac{ds}{d\phi}\right)^2}$, and similarly for η, $d\eta/d\phi$. To get σ, the distance travelled by the projected point $(\xi, \eta, 0)$, we use $d\sigma^2=d\xi^2+d\eta^2$; thus

$$\left(\frac{d\sigma}{d\phi}\right)^2=(l-s)^2\Big/\left(\frac{ds}{d\phi}\right)^4\tan^2\alpha\left[\left(\frac{ds}{d\phi}\right)^2\left(\frac{m^2}{\pi^2}+z^2\right)\right.$$
$$\left.+\frac{ds}{d\phi}\frac{d^2s}{d\phi^2}\left(\frac{m}{\pi}z\right)+\left(\frac{d^2s}{d\phi^2}\right)^2\left(\frac{m^2}{4\pi^2}+z^2\right)\right].$$

Every term in the right member being a known function of ϕ, we denote the entire member by $F(\phi)$, and write

$$\sigma=\int_0^{2n\pi}\sqrt{[F(\phi)]}d\phi.$$

Various reductions are possible in the integrand, but they do not seem to lead to an expression of the indefinite integral.

II. *Remark.* On a "rough" cone the thread may be wound following other curves than geodesics; and with sufficient friction the turns may be wound at uniform distance apart. But even this is impossible if the end of the thread be required to remain in some horizontal plane; so that the proposed problem is impossible (even with friction) if by "trace" we are to understand a curve actually travelled by the end of the thread. Let us prove this.

At every instant the unwound taut thread must lie in the plane tangent to the cone at the point of contact, P; so likewise must the line PT tangent to the curve on the cone at P. But the direction of PT (or of the next succeeding "element of arc") is determined by the intersection of the conical surface with the plane which is determined by the taut thread and the instantaneous direction of motion of its end-point. Since the tangent line PT lies both in the latter plane and in the tangent plane, PT must coincide with the taut thread which lies in these two planes. Thus in winding the thread about the cone, the unwound portion must at every instant be tangent to the curve on the cone.

Now if the end-point (ξ, η, ς) is to remain constantly in a plane at any distance g from the equation of PT, the tangent at (x, y, z),

$$\frac{\xi-x}{\frac{dx}{ds}}=\frac{\eta-y}{\frac{dy}{ds}}=\frac{q-z}{\frac{dz}{ds}}=l-s, \text{ (the length yet unwound).}$$

Separating variables in the last equation, $dz/(q-z)=ds/(l-s)$, whence, p being constant, $p(q-z)=(l-s)$. From this, $p^2dz^2=ds^2=dx^2+dy^2+dz^2$, so that $(p^2-1)dz^2=dx^2+dy^2=\tan^2\alpha(dz^2+z^2d\phi^2)$, since as in (I) above, $r=\tan\alpha.z\cos\phi$, etc. Thus $dz^2(p^2-\sec^2\alpha)=z^2\tan^2\alpha.d\phi^2$, or

$$\frac{dz}{z}=\frac{\pm\tan\alpha}{\sqrt{[p^2-\sec^2\alpha]}}d\phi=\pm Kd\phi, \text{ say,}$$

the upper or lower sign being taken according as the winding proceeds away from or toward the vertex, and the constant K being evidently real for a real curve. The integral $z=A.e\pm K\phi$ shows both that the winding process cannot begin at the vertex, since z vanishes only if $A=0$ in which case $z\equiv0$, and also that the turns are not at a uniform distance apart, since the addition of 2π to ϕ serves merely to multiply z by a constant factor.

III. Suppose now that the end-point (ξ, η, ς) moves so that the turns on the cone ("rough," of course) are equidistant as proposed, and let us find the length of the curve traced by the projection $(\xi, \eta, 0)$.

The requirement of equidistant turns, viz., $z(\phi+2\pi)=z(\phi)+m$, where the constant m is negative if the turns approach the vertex, does not define z as a function of ϕ, but it can be most simply satisfied by taking $z=A+m\phi/2\pi$, A being any constant not zero. Then

$$dx=\tan\alpha\left[\frac{m}{2\pi}\cos\phi-z\sin\phi\right]d\phi,\quad dy=\tan\alpha\left[\frac{m}{2\pi}\sin\phi+z\cos\phi\right]d\phi,$$

where z is written for brevity. Then

$$ds^2=\left[\frac{m^2}{4\pi^2}\sec^2\alpha+\tan^2\alpha\left(A+\frac{m\phi}{2\pi}\right)^2\right]d\phi^2,$$

from which s can be obtained by a simple integration; and in particular the entire length l may be found in terms of n, α and the practically arbitrary A, m. Then to find the length, σ, of the path of $(\xi, \eta, 0)$ we have

$$\xi=x+(l-s)\frac{dx}{d\phi}\Big/\frac{ds}{d\phi},\quad \eta=y+(l-s)\frac{dy}{d\phi}\Big/\frac{ds}{d\phi};$$

$$\frac{d\xi}{d\phi}=(l-s)\cdot\frac{\frac{ds}{d\phi}\frac{d^2x}{d\phi^2}-\frac{d^2s}{d\phi^2}\frac{dx}{d\phi}}{\left(\frac{ds}{d\phi}\right)^2},\quad \frac{d\eta}{d\phi}=(l-s)\frac{\frac{ds}{d\phi}\cdot\frac{d^2y}{d\phi^2}-\frac{d^2s}{d\phi^2}\frac{dy}{d\phi}}{\left(\frac{ds}{d\phi}\right)^2};$$

$$\left(\frac{d\sigma}{d\phi}\right)^2=\left(\frac{d\xi}{d\phi}\right)^2+\left(\frac{d\eta}{d\phi}\right)^2=(l-s)^2(\ \ldots$$

as on bottom of page 107, except that the former m is here $-m$, etc., to former conclusion).

IV. A thread whose turns about a cone have been made uniformly distant by the help of friction may be *unwound* without the end-point retracing its former path; and as the unwound portion need not now be tangent to the curve on the cone, the end-point may travel in a horizontal plane. Let us find the length of its path, again taking arbitrarily $z=A+m\phi/2\pi$. The taut thread must still be in the tangent plane: $(\xi-x)x+(\eta-y)y+(q-z)(-z\tan^2\alpha)=0$; and $(\xi-x)^2+(\eta-y)^2+(\varsigma-z)^2=(l-s)^2$. Now let $\xi-x=(l-s)\cos a$, $\eta-y=(l-s)\cos b$, $q-z=(l-s)\cos c$, introducing the direction-cosines of the

string. Then $\cos^2 a+\cos^2 b=\sin^2 c$, and $\cos a.\cos\phi+\cos b\sin\phi=\cos c\tan\alpha$, (dividing x and y by $z\tan\alpha$). Combining the first of these equations with that obtained by squaring the second, we find $\cos a\sin\phi-\cos b\cos\phi=\pm\sqrt{[1-\cos^2 c\sec^2\alpha]}$. Thus

$$\cos a=(\xi-x)/(l-s)=\pm\sin\phi\sqrt{[1-\cos^2 c\sec^2\alpha]}+\cos c.\tan\alpha.\cos\phi;$$
$$\cos b=(\eta-y)/(l-s)=\mp\cos\phi\sqrt{[1-\cos^2 c\sec^2\alpha]}+\cos c.\tan\alpha.\sin\phi.$$

Since x, y, z, s, $\cos c$ are all known functions of ϕ, these equations furnish $d\xi$ and $d\eta$ in terms of ϕ and $d\phi$, and thus the problem of finding σ is reduced to a quadrature.

NOTES AND NEWS.

Mr. C. H. Forsyth of Michigan University has accepted the chair of mathematics in Eureka College, Eureka, Illinois. M.

Mr. C. A. Barnhart, assistant in the University of Illinois, has accepted the chair of Mathematics in Carthage College, Carthage, Illinois. M.

Dr. R. K. Morley, instructor in the University of Illinois, has accepted an assistant professorship of mathematics in Worcester Polytechnic Institute, Worcester, Mass. M.

Dr. G. F. McEwen, instructor in mathematics in the University of Illinois, has accepted a position in the Marine Biological station of the University of California. This station is located at Lajolla, Cal. M.

A circular, bearing the signature of about seventy prominent mathematicians, calls attention to the fact that during August of the present year Professor Felix Klein of Göttingen, Germany, will reach the fortieth anniversary of his appointment as professor of mathematics in the University of Erlangen. It is proposed to give him some token of the thanks and the good wishes of his fellow mathematicians in view of his great services to mathematical progress, and of his excellent work as a mathematical teacher. Attention is called especially to the great influence of the so-called *Erlanger Programm*, which was translated into English by Professor Haskell, and which was published in the *Bulletin of the New York Mathematical Society*, volume 2 (1893), page 215, under the title, "A Comparative Review of Recent Researches in Geometry." It may be added that Klein is the president of the International Commission on the Teaching of Mathematics, which is expected to report to the International Congress of Mathematicians at its fifth meeting, which is to be held at Cambridge, England, during the coming August. M.